Critical Properties of the Synchronization Transition in Space-Time Chaos

Volker Ahlers and Arkady Pikovsky
Department of Physics, University of Potsdam, Postfach 601553, D-14415 Potsdam, Germany
(Received 24 October 2001; published 7 June 2002)

We study two coupled spatially extended dynamical systems which exhibit space-time chaos. The transition to the synchronized state is treated as a nonequilibrium phase transition, where the average synchronization error is the order parameter. The transition in one-dimensional systems is found to be generically in the universality class of the Kardar-Parisi-Zhang equation with a growth-limiting term ("bounded KPZ"). For systems with very strong nonlinearities in the local dynamics, however, the transition is found to be in the universality class of directed percolation.

DOI: 10.1103/PhysRevLett.88.254101 PACS numbers: 05.45.–a, 05.40.Ca

The synchronization of chaotic systems has been a very active field in nonlinear dynamics since its discovery [1]. It has been observed in lasers, electronic circuits, and chemical reactions [2]. In recent years, the synchronization of spatially extended chaotic systems has attracted particular interest [3]. Experimentally, the most promising systems to observe this phenomenon are optical ones. Indeed, broad-area semiconductor lasers [4] demonstrate space-time chaos (STC). Furthermore, the dynamics of semiconductor lasers with time-delayed optical feedback [5] show many aspects of STC [6]. All of these systems can be optically coupled via mutual light injection (cf. [7]). In another context, symmetry breaking of STC in anisotropic liquid crystals with different diffusion constants for different spatial directions [8] can be also interpreted as the synchronization transition in STC.

From the point of view of a general statistical theory, the synchronization of STC is an example of a nonequilibrium phase transition; accordingly its critical properties attracted particular interest [9,10]. As a generic model of this transition a multiplicative noise partial differential equation (MNPDE) has been proposed [11]. The following studies of this MNPDE [12–15] have revealed nontrivial critical exponents, allowing one to speak of a particular universality class.

In this Letter we characterize the synchronization transition by its critical exponents and relate it to known universality classes, making use of the above mentioned MNPDE. The results are in agreement with recent observations in the context of the synchronization of stochastically driven coupled map lattices [10], which indicates the generality of the synchronization transition. It is worth noting that, although the systems we consider below are purely deterministic, they demonstrate the same statistical properties as stochastic ones; it is another example of validity of statistical description of deterministic STC (cf. [16]).

We first restrict our numerical simulations to systems which are discrete in space (one spatial dimension) and time, i.e., coupled map lattices (CMLs) (see, e.g., [17]). CMLs are well suited for modeling STC, allowing one to perform extensive statistical numerical analysis. Our basic model consists of two coupled CMLs,

$$\begin{pmatrix} u_1(x,t+1) \\ u_2(x,t+1) \end{pmatrix} = \begin{pmatrix} 1-\gamma & \gamma \\ \gamma & 1-\gamma \end{pmatrix} \times \begin{pmatrix} (1+\varepsilon\Delta) f\big(u_1(x,t)\big) \\ (1+\varepsilon\Delta) f\big(u_2(x,t)\big) \end{pmatrix}, \quad (1)$$

with the discrete Laplacian

$$\Delta v(x) = v(x-1) - 2v(x) + v(x+1). \quad (2)$$

Here $x = 1, 2, \ldots, L$ and $t = 0, 1, \ldots$ are the discrete space and time variables (with the system length L), $u_{1,2}$ are the state variables, and the nonlinear function $f(u)$ describes the local dynamics. Periodic boundary conditions, $u_{1,2}(x+L,t) = u_{1,2}(x,t)$, are used throughout. There are two coupling parameters: ε accounts for the coupling inside of each CML and can be seen as a diffusion constant, whereas γ represents the strength of the sitewise interaction between the two CMLs. We choose a fixed value $\varepsilon = 1/3$ for the diffusion constant, and vary the coupling parameter γ to study the synchronization transition.

Complete synchronization of two CMLs is achieved if the synchronization error $w(x,t) = u_1(x,t) - u_2(x,t)$ vanishes at all x for $t \to \infty$, which is equivalent to a vanishing spatial average $\langle |w(x,t)| \rangle_x$. In the unsynchronized state, $\langle |w(x,t)| \rangle_x > 0$. The remaining part of this Letter is devoted to a study of the dependence of the average absolute synchronization error $\langle |w| \rangle$, which is the natural order parameter, on the coupling parameter γ.

To illustrate that there are two different types of the synchronization transition, we present first the results of numerical simulations of the coupling (γ) dependence of the average absolute synchronization error $\langle |w| \rangle_{x,t}$, where the time average is taken after saturation. Using the tent map $f(u) = 1 - 2|u - 1/2|$ $(0 \le u \le 1)$ in Eq. (1), we obtain the synchronization transition shown in Fig. 1(a). The same type of transition is observed for the logistic map $f(u) = 4u(1-u)$ $(0 \le u \le 1)$. We will argue below that the transition for these systems belongs to the universality class of the Kardar-Parisi-Zhang (KPZ) equation

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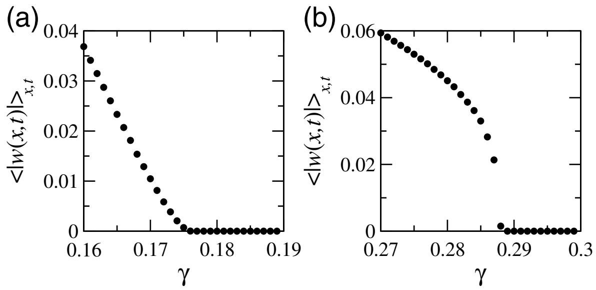

FIG. 1. The synchronization transition for coupled CMLs consisting of (a) tent and (b) Bernoulli maps, respectively (system length $L = 1024$). The shown data are averages over 10 different initial conditions and 5×10^4 iterations (after a transient of 5×10^4 iterations). Similar results are also observed for skew maps, with the slopes of the linear parts being different: a^{-1} and $(1 - a)^{-1}$.

[18] with a saturation term [14]. We therefore denote it as the "bounded KPZ transition" (BKPZ).

We observe, however, a different type of transition if we employ the Bernoulli map $f(u) = 2u \pmod 1$ in Eq. (1); see Fig. 1(b). This transition is continuous like BKPZ, but has a definitely different scaling exponent. Since it will turn out below that the transition for the Bernoulli maps belongs to the universality class of directed percolation (DP, see, e.g., Ref. [14]), we refer to it as the "DP transition."

The crucial difference between systems showing each of the two transition types is the presence or absence of a strong nonlinearity (such as a discontinuity) in the local map: the (skew) tent and logistic maps are continuous, while the (skew) Bernoulli maps are discontinuous. The same observations were recently reported for spatially extended systems which are not coupled to each other, but driven by a common noise process [10]. The authors of Ref. [10] attribute the DP transition to an instability with respect to finite perturbations of linearly stable systems.

The different nature of the two transitions is visualized in Fig. 2, which shows the spatiotemporal evolution of the

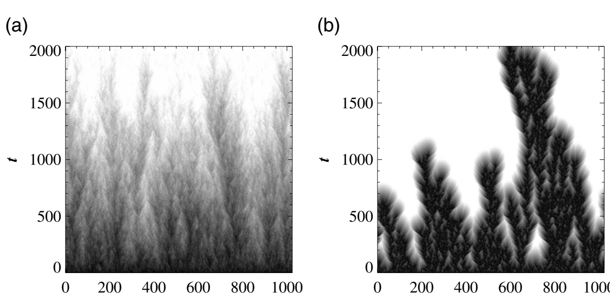

FIG. 2. Spatiotemporal evolution of the synchronization error of coupled CMLs ($L = 1024$) with coupling parameters γ slightly larger than the critical values γ_c: gray scale plot of the absolute synchronization error $|w(x,t)|$ (darker shades correspond to larger values). (a) Tent CMLs, $\gamma = 0.181$; (b) Bernoulli CMLs, $\gamma = 0.291$. The coupling is switched on at $t = 0$ after some transient evolution of the uncoupled CMLs (random initial conditions).

absolute synchronization error $|w(x,t)|$ for CMLs consisting of (a) tent and (b) Bernoulli maps, respectively, for values of γ slightly larger than the critical γ_c.

We now turn our attention towards the temporal evolution of a small perturbation $w(x,t)$ of the synchronized state $u_1(x,t) = u_2(x,t)$, which in the linear order obeys the equation

$$w(x, t + 1) = (1 - 2\gamma)(1 + \varepsilon\Delta)[f'(u_1(x,t))w(x,t)]. \tag{3}$$

The exponential growth rate of the norm $\|w\|$ is known as the transverse Lyapunov exponent,

$$\lambda_\perp = \lim_{t\to\infty}\frac{1}{t}\ln\|w(t)\| = \ln(1 - 2\gamma) + \Lambda, \tag{4}$$

where Λ is the usual Lyapunov exponent of a single CML. The synchronized state is linearly stable if $\lambda_\perp < 0$, i.e., if $2\gamma > 2\gamma_c = 1 - e^{-\Lambda}$ (see [1]).

A numerical calculation of the Lyapunov exponent reveals that for CMLs consisting of continuous maps $\langle|w|\rangle_{x,t}$ vanishes exactly at the above defined critical value γ_c [we obtained, e.g., for the tent and the logistic map $\gamma_c = 0.1760$ and $\gamma_c = 0.1584$ basing on (4), while direct modelling gave 0.17605(5) and 0.1584(1), correspondingly]. For CMLs consisting of discontinuous maps there exists a range of γ, for which $\langle|w|\rangle_{x,t} > 0$ although $\lambda_\perp$ is definitely negative. In particular, for the Bernoulli map with $a = 1/2$ and $a = 1/3$ the values of γ_c following from (4) are 0.250 and 0.2430, while the transition occurs at 0.2875(1) and 0.2800(5), correspondingly. In other words, no synchronization is observed in this range of γ although the synchronized state is linearly stable. This behavior, the instability with respect to finite perturbations of linearly stable spatially extended systems with strong nonlinearities in the local dynamics, is known for systems demonstrating "stable chaos" [19]. In the following, we will demonstrate that the transition observed for continuous local maps, at which the transverse Lyapunov exponent changes sign, is of the BKPZ type, while the transition mediated by instability to finite perturbations is of the DP type.

Next we report numerical findings of the scaling indices of the BKPZ transition. The main indices δ and β appear in the dependencies of the order parameter: $\langle|w|\rangle_x \sim t^{-\delta}$ for $\gamma = \gamma_c$ and $\langle|w|\rangle_{x,t} \sim (\gamma_c - \gamma)^\beta$ for $\gamma_c \gtrsim \gamma$. First, for a CML consisting of $L = 2^{20}$ tent maps, we estimated the critical coupling parameter by two methods. By extrapolating the values $\Lambda(L)$ to $L \to \infty$, using the relation [20,21] $\Lambda(L) - \Lambda(\infty) \sim L^{-1}$, we obtained $\gamma_c = 0.176\,14$. A very close value $\gamma_c = 0.176\,15 \pm 0.000\,05$ was obtained by looking for the best power-law behavior of $\langle|w|\rangle_x(t)$ for different values of γ; see Fig. 3(a). A scaling analysis of this graph gives the exponent $\delta = 1.26 \pm 0.03$. Second, we studied the scaling behavior of the data shown in Fig. 1(a) for a CML consisting of $L = 1024$ tent maps. Using the critical value $\gamma_c = 0.1760$, we obtain

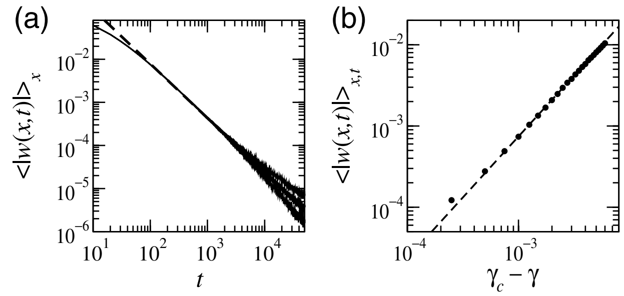

FIG. 3. Coupled tent CMLs: (a) Time dependence of the space-averaged absolute synchronization error, $\langle|w|\rangle_x(t)$, for coupling parameters $\gamma \in \{0.1761, 0.17615, 0.1762\}$ (upper to lower solid lines, averaged over five initial conditions, $L = 2^{20}$); the dashed line has a slope -1.26, as obtained from a scaling analysis. (b) Coupling dependence of $\langle|w|\rangle_{x,t}$ (time averaged after saturation) with $\gamma_c = 0.1760$ ($L = 1024$); the dashed line has a slope 1.5.

$\beta = 1.50 \pm 0.05$; see Fig. 3(b). A better estimate of β, however, is difficult due to the uncertainty in the knowledge of γ_c.

The critical indices of the synchronization transition are to be compared with those of the other models of the BKPZ transition. We introduce first a multiplicative noise partial differential equation with a cubic saturation term as a model of the perturbation dynamics. The linear dynamics of the perturbations [Eq. (3)] account for the exponential growth with finite-time fluctuations of the growth rate as well as spatial diffusion; see Refs. [11,20]. Adding the nonlinear saturation term ensures that the synchronization error $w(x,t)$ remains bounded [14]. We thus obtain

$$\frac{\partial w(x,t)}{\partial t} = [a + \xi(x,t) - p|w(x,t)|^2]w(x,t) + \varepsilon \frac{\partial^2 w(x,t)}{\partial x^2}. \tag{5}$$

The control parameter a is connected with the transverse Lyapunov exponent $\lambda_\perp$, which depends on γ according to Eq. (4); the critical point a_c corresponds to the value γ_c at which the transverse Lyapunov exponent crosses zero. The Gaussian stochastic process $\xi(x,t)$ has the properties

$$\langle \xi(x,t)\rangle = 0,$$
$$\langle \xi(x,t)\xi(x',t')\rangle = 2\sigma^2\delta(x-x')\delta(t-t').$$

By application of the Hopf-Cole transformation, $h = \ln|w|$, Eq. (5) can be transformed into [11]

$$\partial h(x,t)/\partial t = a + \xi(x,t) - pe^{2h(x,t)} + \varepsilon\partial^2 h(x,t)/\partial x^2 + \varepsilon[\partial h(x,t)/\partial x]^2, \tag{6}$$

which is the KPZ equation [18] with an additional saturation term. In Eq. (6), synchronization corresponds to an interface moving towards $-\infty$, and the exponential saturation term prevents the interface from moving towards large positive values; thus the transition can be termed as "bounded KPZ." The negative average interface velocity gives the transverse Lyapunov exponent $\lambda_\perp$.

The continuous phase transition in Eq. (5) that occurs in dependence on a has been studied by numerical and renormalization group methods [12,13,15]. However, the scaling indices are not known with a very good accuracy, in particular due to the difficulty of estimating a_c. In the literature, the values $\delta = 1.10 \pm 0.05$ and $\beta = 1.70 \pm 0.05$ [13] (as well as $\delta = 1.10 \pm 0.12$ and $\beta = 1.50 \pm 0.15$ [15]) have been reported for the multiplicative noise equation (5). From a numerical simulation of Eq. (6) by means of a discrete growth model with a limiting wall where the critical value a_c is known exactly, one obtains $\delta = 1.17 \pm 0.05$ and $\beta = 1.70 \pm 0.05$ (details of this analysis will be reported elsewhere [22]). The most probable reason for the small discrepancy between these values and our findings for the coupled CMLs (see above) is the presence of spatiotemporal correlations in the chaotic dynamics. Another possible cause for the discrepancy may be the difference in the updating in the CMLs and stochastic models (parallel vs asynchronous); it is known that this is crucial for the values of critical indices at other transitions in STC [23].

For a CML consisting of Bernoulli maps, the scaling of $\langle|w|\rangle_x(t)$ at $\gamma_c = 0.2875$ agrees well with the value of the critical exponent, $\delta = 0.159$, that is known for DP [24] (data not shown here). Furthermore, the finite-size scaling relation

$$\langle|w|\rangle_x(t) \sim L^{-\beta/\nu_\perp} f(t/L^z)$$

known for DP [23] is very well fulfilled by the coupled Bernoulli CMLs; see Fig. 4. As we already mentioned above, the transition for CMLs consisting of discontinuous maps is not ruled by linear stability properties. Thus the stochastic model (5), which is based on the linear perturbation dynamics, does not apply to such systems. In Fig. 2(b) it is obvious that for coupled Bernoulli CMLs there are no desynchronization events in already synchronized regions. Thus the synchronized state is absorbing (as was

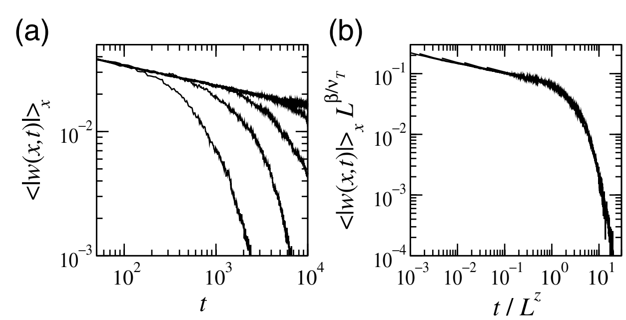

FIG. 4. Coupled Bernoulli CMLs: finite-size scaling at the critical coupling parameter $\gamma_c = 0.2875$. $\langle|w|\rangle_x(t)$ is plotted in (a) unscaled and (b) scaled coordinates for system lengths $L \in \{32, 64, 128, 256, 512, 1024\}$ (lower to upper lines). The DP values $\beta/\nu_\perp = 0.252$ and $z = 1.581$ [24] are used in (b).

already argued in Ref. [10], see also Ref. [9]). The existence of an absorbing state is a prerequisite of directed percolation [14].

We are of the opinion that the BKPZ transition is the typical synchronization transition. The DP transition was only observed for CMLs consisting of maps with strong nonlinearities (see also Ref. [10]); it is not clear if it can occur in oscillator lattices or PDEs. Since the stochastic PDE model, Eq. (5), is not limited to discrete systems (such as CMLs), one can expect to observe the BKPZ transition also for coupled chaotic PDEs. To check this we have investigated numerically the synchronization transition in two coupled Kuramoto-Sivashinsky (KS) equations (they describe waves on falling liquid films and instabilities of combustion fronts; see [16] for details)

$$\frac{\partial u_{1,2}}{\partial t} + \frac{\partial^2 u_{1,2}}{\partial x^2} + \frac{\partial^4 u_{1,2}}{\partial x^4} + u_1 \frac{\partial u_{1,2}}{\partial x} = \gamma(u_{2,1} - u_{1,2}).$$

In a large spatial domain with periodic boundary conditions a single KS equation demonstrates space-time chaos with the largest Lyapunov exponent $\Lambda \approx 0.0475$. We have studied scaling properties of the synchronization error $||u_1(x,t) - u_2(x,t)||$ near the threshold $\gamma_c = \Lambda$ in a way shown in Fig. 3 above, and were able to estimate the scaling exponents: $\delta = 1.2 \pm 0.1$, $\beta = 1.5 \pm 0.1$ (the uncertainty is, of course, larger than for CMLs because it is difficult to achieve for PDEs the same statistics). These numbers are in agreement with those obtained for CMLs, thus confirming that the synchronization transition in coupled Kuramoto-Sivashinsky equation belongs to the bounded KPZ class. Furthermore, the same transition is found for unidirectionally coupled spatially extended systems. The found generality of the transition suggests that it can be also discovered in equations describing STC in optical systems [4,5].

In conclusion, we have given numerical evidence that the synchronization transition of coupled CMLs consisting of continuous maps belongs to the universality class of the bounded Kardar-Parisi-Zhang (BKPZ) equation. For coupled CMLs consisting of maps with strong nonlinearities (e.g., discontinuous maps), the synchronization transition belongs to the universality class of directed percolation (DP). This can be understood from the observation that in such systems the synchronized state can be unstable with respect to finite perturbations, even if it is linearly stable. Numerical results for coupled KS equations indicate that the synchronization transition in coupled PDEs belongs to the BKPZ universality class. Thus, we expect a strong spatiotemporal intermittency to be generally observed at the synchronization transition of STC. Quantitatively, its features are described by scaling indices β, δ.

While there exist very accurate estimates of the critical exponents of DP, definite values for the BKPZ transition are still missing. A possible remedy is given by a discrete KPZ growth model with a limiting wall, which allows very efficient numerical simulations. Details of this approach will be reported elsewhere [22].

We thank F. Ginelli, P. Grassberger, R. Livi, A. Politi, and A. Torcini for many fruitful discussions and the Deutsche Forschungsgemeinschaft (SFB 555) as well as NATO (Grant No. 973054) for financial support.

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